29 July 2002

PHYSICS LETTERS A

ELSEVIER

Physics Letters A 300 (2002) 221–227

www.elsevier.com/locate/pla

An extended network model with a packet diffusion process

C. Pigorsch, S. Trimper *

Fachbereich Physik, Martin-Luther-Universität, D-06099 Halle, Germany

Received 13 September 2001; received in revised form 18 February 2002; accepted 7 June 2002

Communicated by A.P. Fordy

Abstract

The dynamics of packets diffusion within a self-organized network is analytically studied by means of an extended f-spin kinetic Ising model (Fredrickson–Andersen model) using a Fock-space representation for the master equation. To map the three component system (active, passive and packet cells) onto a lattice we apply two types of second quantized operators. The active cells correspond to mobile states whereas the passive cells correspond to immobile states of the Fredrickson–Andersen model. An inherent cooperativity is included by assuming that the local dynamics and subsequently the local mobilities are restricted by the occupation of neighboring cells. Depending on a temperature-like parameter h^{-1} (interconnectivity) the diffusion of the packet (information) can be almost stopped. Thus we get a separation of the time regimes and transient localization for the intermediate range at low interconnectivity. © 2002 Elsevier Science B.V. All rights reserved.

PACS: 02.50.Ey; 05.20.Dd; 82.20.Mj; 05.50.+q

During the last years there is a huge effort to understand the kinetics of non-equilibrium phenomena. A wide range of discrete and continuous models for such processes is analytically and numerically studied. The problems of interest in this context concern the crystal growth, transport (traffic) models, diffusion processes and supercooled liquids [1–6]. Here, we will apply the kinetics of the Fredrickson–Andersen model (FAM) recently discussed in the framework of the glass transition and related phenomena [5–12]. But we show that this model may be used on other fields like stock trading, citation networks, company relations or internet communications as well [13–16]. In general, we study the diffusion of information within a network system of active links and passive/active cells (or nodes). The switch between a passive and active cell is controlled by the interconnectivity parameter h^{-1}. At maximum interconnectivity the system possesses equal numbers of active and passive cells whereas at the minimum interconnectivity there are only passive cells. Furthermore, the alteration is also controlled by the nearest neighbors. If enough adjacent active cells exist (more than a fixed number f) the active cell can become passive and vice versa. Thus, only a sufficiently active environment may determine and alter the state of a cell as in a citation community where only accepted (active) people may decide about the worth of an opinion of a member in a related field. To this network formed by passive/active cells and active links, consisting of two adjacent active cells, we add further particles which may be assigned to (information) packets. These packets can only diffuse along ac-

* Corresponding author.
E-mail address: trimper@physik.uni-halle.de (S. Trimper).

PII: S0375-9601(02)00806-X

tive links but are confined by passive cells. Therefore, we have diffusion in a self-organized network where information may stick (and therefore be localized) at passive cells or run through a network of active links. This is the same situation in the network where passive routers cannot transfer data but data on an active router should be directed to the next active router. In the present Letter we demonstrate how one may relate this network to the FAM originally formulated for supercooled liquids where the analysis is based on a mesoscopic formulation. Hence, one reduces for supercooled liquid the degrees of freedom to a smaller set of relevant observables. However, in the case of our network model we will immediately start on the mesoscopic scale (e.g., on the elementary scale of routers where the macroscopic scale is the total network). The cell structure enables us to attach to each cell j a local activity m_j where the passive state is realized in case of $m_j < \bar{m}$ and the active one if $m_j > \bar{m}$. The quantity $\bar{m}$ is a fixed threshold of the system. The threshold is chosen so that all cells are passive at zero interconnectivity. The number of cells in the active and the passive state, respectively, need not be conserved. Thus, the real kinetics is more Glauber-like (non-conserved) dynamics. Hence we suppose that the basic dynamics is a flip process between the active and the passive state. It is controlled by self-induced topological restrictions introduced below. This type of dynamics leads to a relaxation behavior resembling to that of a cooperative system. In particular, an elementary switch at a given cell is allowed only if the number of nearest neighbor active cells is equal to or larger than a certain number f, with $0 \leqslant f \leqslant z$ (z is the coordination number of the lattice). Hence, elementary flip processes combined with these restrictions may lead to cooperative arrangements within the underlying network. A broad variety of Fredrickson–Andersen models (or f-spin kinetic Ising models) [7–9] has been studied analytically [5,6,17] and numerically [10–12,18]. The FAM can be classified as an Ising-like model for which the kinetics is limited by restrictions on the ordering of nearest neighbors to a given lattice cell. These self-adapting environments especially influence the long-time behavior of the state relaxation [6,11]. Additionally, here we add packets to the system for which diffusive dynamics are coupled with the existence of the active states. Therefore, we require that active cells are necessary for the motion of packets because passive states block packet diffusion. In other words, data are fixed and static as long as they are assigned to passive cells, but information may diffuse through a network of active cells. In the present work we will incorporate this feature into the Fock-space representation of the master equation in order to compute the packet concentration in a continuous mean-field approximation.

First, we give a short survey about the Fock-space method (known as the quantum Hamiltonian method—more details can be found, e.g., in [3]). A given state in the lattice system can be characterized by a set of discrete numbers $\vec{n} = \{n_i\}$ (or respectively $\vec{v} = \{v_i\}$), where $n_i, v_i \in \{0, 1\}$ denote the local state of a lattice cell i. Furthermore, the following convention is used: $n_i = 1$ (0) refers to the active (passive) state. The state $v_i = 1$ (0) corresponds to a cell i occupied (non-occupied) by a packet. We start from the one-step master equation

$$\partial_t P(\vec{n}, \vec{v}, t) = L' P(\vec{n}, \vec{v}, t), \tag{1}$$

where $P(\vec{n}, \vec{v}, t)$ is the probability for a certain configuration $\{\vec{n}, \vec{v}\}$. The linear operator L', specified by the dynamics of the system, describes the time evolution. Then, following [19,20], the probabilities $P(\vec{n}, \vec{v}, t)$ can be related to the Fock-space state vector $|F(t)\rangle$ as a weight relative to the decomposition into the basis vectors $|\vec{n}\rangle \otimes |\vec{v}\rangle$ of an orthonormal vector space,

$$|F(t)\rangle = \sum_{\vec{n},\vec{v}} P(\vec{n}, \vec{v}, t) |\vec{n}\rangle \otimes |\vec{v}\rangle. \tag{2}$$

This equation leads to the quantum formulation of the master equation resulting in

$$\partial_t |F(t)\rangle = \hat{L} |F(t)\rangle, \tag{3}$$

where L' corresponds to the operator $\hat{L}$ in the Fock-space representation. This procedure was originally derived for Bose-like systems [19,20] and was later applied to Fermi-like systems [4,21,22]. Recently, we proposed a further extension, applying para-Fermi statistics for differently restricted occupation numbers [6,23]. The average of a physical quantity $G(\vec{n}, \vec{v})$ is given by the trace over $\hat{G}$:

$$\langle \hat{G}(t) \rangle = \sum_{\vec{n},\vec{v}} P(\vec{n}, \vec{v}, t) G(\vec{n}, \vec{v}) = \langle \vec{r} | \hat{G} | F(t) \rangle, \tag{4}$$

where $\langle\vec{r}|$ is the reference state related to the basis

$$\langle\vec{r}| = \sum_{\vec{n},\vec{v}} \langle\vec{n}| \otimes \langle\vec{v}| = \bigotimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{5}$$

The reference state is completely determined by the basis $\{|\vec{n}\rangle, |\vec{v}\rangle\}$ of the Fock space and does not depend on the particular model or the evolution operator $\hat{L}$. The conservation of total probability is manifested by $\langle\vec{r}|\hat{L} = 0$. Therefore, the equation of motion is given by

$$\partial_t \langle \hat{G}(t) \rangle = \langle \vec{r} | \hat{G}\hat{L} | F(t) \rangle = \langle \vec{r} | [\hat{G}, \hat{L}]_- | F(t) \rangle. \tag{6}$$

Notice that this dynamical equation depends on both the algebraic properties of the underlying operators and the mathematical structure of $\hat{L}$. Next, we introduce the second quantized lowering $a_i(v_i)$ and raising $a_i^\dagger(v_i^\dagger)$ operators, forming the evolution operator $\hat{L}$, to create the basis states $|\vec{n}\rangle$ ($|\vec{v}\rangle$) from the vacuum state $|0\rangle$. Both types of (independent of each other and hence commuting) operators fulfill the relationship (for Paulions)

$$a_i a_j^\dagger + a_j^\dagger a_i = \delta_{i,j} + 2a_j^\dagger a_i (1 - \delta_{i,j}). \tag{7}$$

As mentioned above, the inherent properties of the FAM is the restriction, on the flip dynamics at cell i, $\sigma_i \leftrightarrow -\sigma_i$,

$$\frac{1}{2} \sum_{j(i)} \langle n_j | (1 + \sigma_j) | n_j \rangle = \sum_{j(i)} \langle n_j | \hat{A}_j | n_j \rangle \geqslant f, \tag{8}$$

where $j(i)$ denotes the sum over all adjacent cells of i and f is the restriction number. The number operators $\hat{A}_j$ and $\hat{V}_j$ denote $a_j^\dagger a_j$ and $v_j^\dagger v_j$ as usual. Concerning the motion of information packets we postulate diffusive motion of the particles coupled to the existence of active cells at the initial and final sites. This exchange process, due to Kawasaki, enhances the mobility of packets in active neighborhoods whereas it slows it down inside a passive cluster. Summarizing, we consider the evolution operators taking the form

$$\begin{aligned} L &= L_F + L_E, \\ L_F &= + \sum_i \lambda_{BA} (1 - a_i) a_i^\dagger \sum_{\langle m_1 \ldots m_f, i \rangle} \hat{A}_{m_1} \ldots \hat{A}_{m_f} \\ &\quad + \sum_i \lambda_{AB} (1 - a_i^\dagger) a_i \sum_{\langle m_1 \ldots m_f, i \rangle} \hat{A}_{m_1} \ldots \hat{A}_{m_f}, \\ L_E &= + \sum_{\langle rs \rangle} D_0 [v_r^\dagger v_s - (1 - \hat{V}_r) \hat{V}_s] \hat{A}_r \hat{A}_s \\ &\quad + \text{symmetric term}, \end{aligned} \tag{9}$$

where the quantities $\lambda_{AB} = \tilde{\lambda} \exp[h]$, $\lambda_{BA} = \tilde{\lambda} \times \exp[-h]$ and D_0 are the kinetic coefficients for the diffusion process. They are appropriately thermodynamically weighted to fulfill the detailed balance condition. As mentioned before the parameter h corresponds to the inverse interconnectivity between cells. The higher h is set the lower is the interconnectivity of the network. The first term of L_F reflects the flip process from the passive to the active state whereas the second term represents the inverse process. The second part, L_E, expresses the exchange process $V_i + V_j \leftrightarrow V_j + V_i$ related to the existence of an active link between adjacent cells. The term

$$\sum_{\langle m_1 \ldots m_f, i \rangle} \hat{A}_{m_1} \ldots \hat{A}_{m_f} \tag{10}$$

in Eq. (9) represents the kinetic restriction mentioned above. The abbreviation $\langle m_1 \ldots m_f, i \rangle$ denotes the sets of all the f lattice cells neighboring to the cell i. The operator $\hat{A}_m$ yields a non-zero value only if the cell m is active, so that expression (10) differs from zero if it is applied to a cell surrounded by at least f active cells. Using Eq. (4) the temporal evolution of the two relevant observables $\langle \hat{A}_k \rangle$ and $\langle \hat{V}_k \rangle$ results in

$$\begin{aligned} \partial_t \langle \hat{A}_k \rangle &= \lambda_{BA} \sum_{\langle m_1 \ldots m_f, k \rangle} \langle \hat{B}_k \hat{A}_{m_1} \ldots \hat{A}_{m_f} \rangle \\ &\quad - \lambda_{AB} \sum_{\langle m_1 \ldots m_f, k \rangle} \langle \hat{A}_k \hat{A}_{m_1} \ldots \hat{A}_{m_f} \rangle, \\ \partial_t \langle \hat{V}_k \rangle &= 2D \langle \nabla_k (\hat{A}_k^2 \nabla_k \hat{V}_k) \rangle, \end{aligned} \tag{11}$$

where we exploit the discrete form of the Laplacian

$$\Delta_k O_k = \frac{1}{l^2} \sum_{r(k)} (O_r - O_k). \tag{12}$$

The diffusion coefficient is modified to $D = D_0 l^2$, where l is the length of the lattice cell. The current for the diffusive motion of the packets is given by

$$j = -2D \hat{A}_k^2 \nabla_k \hat{V}_k. \tag{13}$$

It is intuitively obvious that the effective diffusion coefficient should depend on the squared concentration of the active cells, and therefore on the number of active links.

Now we consider a solution of the hierarchy of equations by decoupling all average values in a mean-field approximation. Such an approach seems to be justified because we are interested in the long time limit, whereas all the elementary processes are realized on a more microscopic scale. Due to the fact that the formula for the vacancies concerns a single lattice index, we may neglect it. Then, we obtain for the evolution equation of the packets

$$\partial_t \langle V(t)\rangle = 2D\nabla\big[\langle A(t)\rangle^2 \nabla \langle V(t)\rangle\big], \tag{14}$$

whereas the equation of motion for the active state yields

$$\partial_t \langle A(t)\rangle = \lambda_{BA}\zeta\langle 1 - A(t)\rangle\langle A(t)\rangle^f - \lambda_{AB}\zeta\langle A(t)\rangle^{f+1}. \tag{15}$$

The temporal solution of Eq. (15) is easily found to be

$$\langle A(t)\rangle = \bar{A} + \big[A(0) - \bar{A}\big]\exp\left(-\frac{t}{\tau_1}\right) \tag{16}$$

with the initial value $A(0)$ and the steady state solution

$$\bar{A} = \frac{\lambda_{BA}}{\lambda_{AB} + \lambda_{BA}} = \frac{1}{\exp(2h) + 1}. \tag{17}$$

As the inverse relaxation time of the flip process we find

$$\tau_1^{-1} = (\lambda_{BA} + \lambda_{AB})\zeta\bar{A}^f \tag{18}$$

with $\zeta = z \cdot \ldots \cdot (f+1)$. Notice that the steady state solution in the mean-field approximation is the same as the solution for the paramagnetic lattice gas and is independent of f. In contrast, the relaxation time depends on $\bar{A}^f$. Because the dynamics of the packets is globally conserved the steady solution is fixed for all time by the initial distribution, i.e.,

$$\bar{V} = \frac{\int V(x,0)\,dx}{\int dx}. \tag{19}$$

Making a linear stability analysis we get from Eqs. (14) and (15)

$$\partial_t \begin{pmatrix} \delta A(\vec{q},t) \\ \delta V(\vec{q},t) \end{pmatrix} = -\begin{pmatrix} \tau_1^{-1} & 0 \\ 0 & 2D\bar{A}^2 q^2 \end{pmatrix}\begin{pmatrix} \delta A(\vec{q},t) \\ \delta V(\vec{q},t) \end{pmatrix}, \tag{20}$$

where $\delta A(\vec{q},t)$ and $\delta V(\vec{q},t)$ are the Fourier transformed small fluctuations around the steady-state values $\bar{A}$ and $\bar{V}$. The divergence of the second relaxation time $\tau_2^{-1} = 2D\bar{A}^2 q^2$ at the wave number $\vec{q} = \vec{0}$ reflects the global conservation of the information in our network. Due to the quadratic dependence on $\bar{A}$ the relaxation time τ_2 rapidly increases if the active cells become passive. Obviously, the steady state is stable against perturbations, as indicated by the negative sign. To gain more insight into the diffusion process for information associated with packets we consider its evolution equation (14) by applying the mean-field solution as for the active state (16). This leads to a spatial-independent, but time-dependent effective diffusion coefficient with

$$\partial_t \langle V(t)\rangle = 2D\left[\bar{A} + \big(A(0) - \bar{A}\big)\exp\left(-\frac{t}{\tau_1}\right)\right]^2 \times \nabla^2 \langle V(t)\rangle. \tag{21}$$

A measure for the fluctuations of the packet concentration is given by

$$\begin{aligned} F(t) &= \int_0^t \langle A(t')\rangle^2\, dt \\ &= \bar{A}^2 t + 2\tau_1 \bar{A}\big(A(0) - \bar{A}\big)\big(1 - e^{-t/\tau_1}\big) \\ &\quad + \frac{\tau_1}{2}\big(A(0) - \bar{A}\big)^2\big(1 - e^{-2t/\tau_1}\big). \end{aligned} \tag{22}$$

Studying the asymptotic limits of $F(t)$ we recognize different temporal regimes. Whereas for small times the fluctuations are dominated by the initial values of the active cells $F(t) \sim A^2(0)t$, the fluctuations are approximated by $F(t) \sim \bar{A}^2 t$ for long times. Although the fluctuations go to infinity in the long-time limit (in this mean-field theory) there is transient localization in an intermediate temporal range especially for small interconnectivity, which is depicted in Fig. 1. Only a small portion of cells is active ($\bar{A} \ll (A(0) - \bar{A})$) inhibiting the diffusion process. The exponential functions in the second and the third terms in Eq. (22) are negligible so that the fluctuation $F(t)$ remains almost constant. To see if there is transient localization one must compare the relaxation time τ in Eq. (18) and the time scale t_L, where the first term is equal the second, and the third term in Eq. (22). In this connection, we assume that the exponential functions may be neglected. The time t_L associated with this point is (sup-

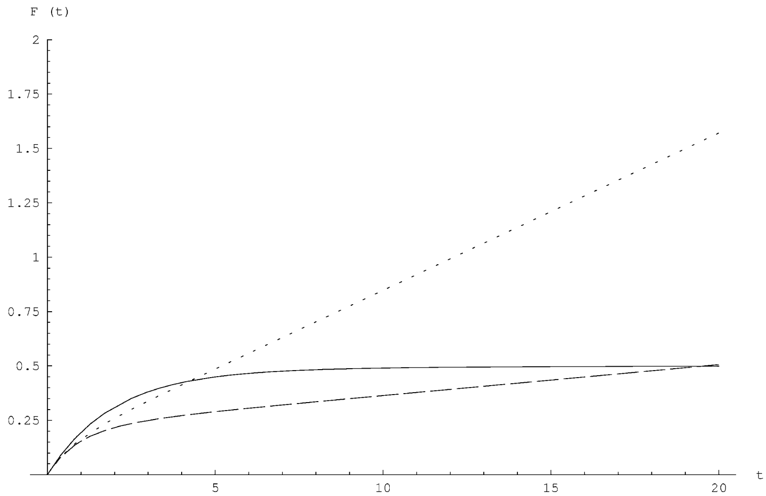

Fig. 1. The fluctuation measure $F(t)$ for the interconnectivities $h^{-1} = 0.5$ (solid line), 1 (dashed line) and 2 (dotted line) as function of the time t ($D = 1$, $f = 1$ and $A(0) = 1/2$).

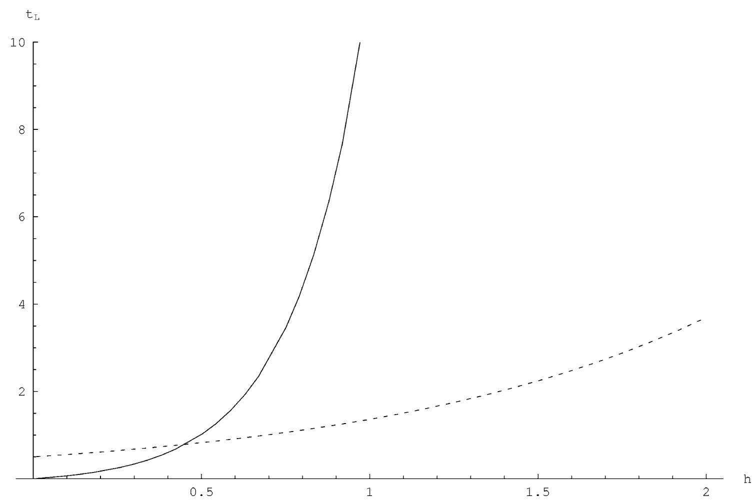

Fig. 2. The transition time t_L in comparison to the relaxation time τ_1 as functions of the inverse interconnectivity h ($D = 1$, $f = 1$ and $A(0) = 1/2$).

posing $A(0) > \bar{A}$)

$$t_L = \frac{\tau_1}{2}\left[\left(\frac{A(0)}{\bar{A}}\right)^2 - 1\right]. \tag{23}$$

Obviously, the crossover to localization ($t_L \gg \tau_1$) becomes possible only if the concentration $\bar{A}$ is small enough (sufficient low interconnectivity) over the time interval $\tau_1 < t < t_L$, see Fig. 2. A rough estimation yields the ratio

$$\frac{t_L}{\tau_1} \sim \exp(4h). \tag{24}$$

Further, we may use $F(t)$ to transform the partial time derivative in the evolution equation

$$\frac{\partial}{\partial t} = \frac{dF(t)}{dt}\frac{\partial}{\partial F(t)} = \langle A(t)\rangle^2 \frac{\partial}{\partial F(t)}, \tag{25}$$

and hence obtain the ordinary diffusion equation

$$\frac{\partial\langle V(t)\rangle}{\partial F(t)} = 2D\nabla^2\langle V(t)\rangle$$

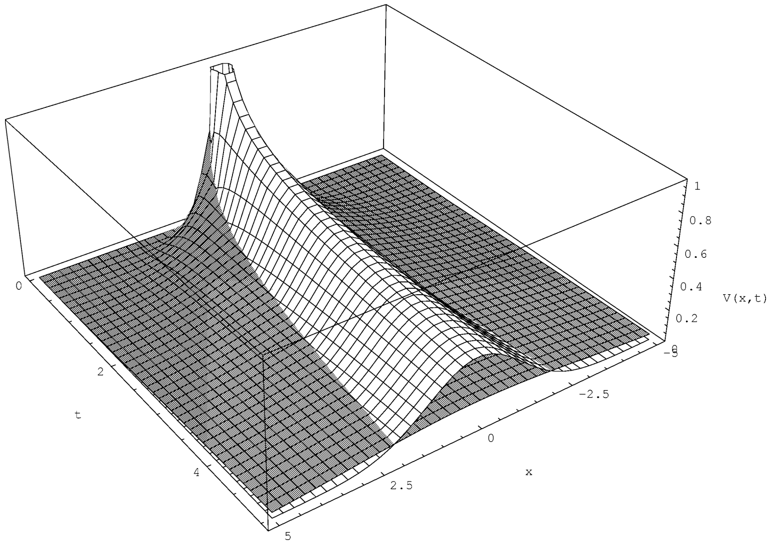

Fig. 3. The diffusion of the packages concentration in the restricted network model (upper "localized" white graph) in comparison to the ordinary diffusion process (lower gray graph) embedded in one dimension ($D = 1$, $f = 1$ and $A(0) = 1/2$).

by applying the transformed Eq. (14)

$$\frac{\partial\langle V(t)\rangle}{\partial t} = \langle A(t)\rangle^2 \frac{\partial\langle V(t)\rangle}{\partial F(t)} = 2D\langle A(t)\rangle^2 \nabla^2 \langle V(t)\rangle. \tag{26}$$

If we now start with a δ-distributed information (packet) density, i.e., place the information at one point and observe how it will be distributed in time,

$$V(\vec{x}, 0) = \bar{V}\delta(\vec{x}), \tag{27}$$

we get the result of Eq. (14):

$$V(\vec{x}, t) = \frac{1}{(4\pi DF(t))^{d/2}} \exp\left(\frac{\vec{x}^2}{4\pi DF(t)}\right). \tag{28}$$

Hence, the behavior of $F(t)$ directly influences the diffusion of the information. If $F(t)$ remains almost constant in time the diffusion process and therefore the distribution of the information stops, and the data packets are localized, see Fig. 3.

In our extended kinetic Ising model we find transient localization due to the coupling of the diffusive dynamics for data packets with the existence of active cells at sufficiently low connectivity. Because there are small portions of the cells in this case, the diffusion almost stops. Therefore, the packets are fixed at or near their initial position, and information cannot be widespread. Thus, we may distinguish two time regimes. First of all the fast active–passive state relaxation takes place. The higher the restriction number f the more the relaxation time slows down (compare Eq. (18)). Then after a while, the influence of the (slow) diffusion is effective and it equilibrates the packet concentration in the total system. However, for high enough interconnectivity there are enough active cells so that the diffusion can speed the equilibration and information can easily spread through the network. The mean field approximation for the Fredrickson–Andersen model [24] may provide false results. The mean-field solution leads to a dynamics which completely breaks down below a critical interconnectivity. That means below a critical interconnectivity information would spread through a fixed network like through a sponge (Bond percolation results would apply to this case; see, e.g., [25]). Here, we exploit a more sophisticated approximation, taking local processes into account. But we expect that the transient localization found leads to permanent localization at low interconnectivity if $f > z$. In this case stable clusters of passive cells exist at any interconnectivity in contrast to the case $f \leqslant z$ where all cells can be activated [24]. To prove this, we refer the reader to a theorem of van Enter [26] describing a bootstrap percolation model [27] (or diffusion percolation [28]).

Acknowledgements

We thank Michael Schulz for several discussions. We are very grateful to the Deutsche Forschungsgemeinschaft (DFG-SFB 418) for the financial support.

References

[1] C. Godrèche (Ed.), Solids Far from Equilibrium, Cambridge University Press, Cambridge, 1992.
[2] V. Privman (Ed.), Nonequilibrium Statistical Mechanics in One Dimension, Cambridge University Press, Cambridge, 1997.
[3] D. Mattis, M.L. Glasser, Rev. Mod. Phys. 70 (1998) 979.
[4] G.M. Schütz, in: C. Domb, J. Lebowitz (Eds.), Exactly Solvable Models for Many-Body Systems Far from Equilibrium, Phase Transitions and Critical Phenomena, Academic Press, London, 2000.
[5] M. Schulz, S. Trimper, J. Stat. Phys. 94 (1999) 173.
[6] C. Pigorsch, M. Schulz, S. Trimper, Int. J. Mod. Phys. B 13 (1999) 1379.
[7] G.H. Fredrickson, H.C. Andersen, Phys. Rev. Lett. 53 (1984) 1244.
[8] G.H. Fredrickson, H.C. Andersen, J. Chem. Phys. 84 (1985) 5822.
[9] G.H. Fredrickson, Ann. Rev. Phys. Chem. 39 (1988) 149.
[10] S. Butler, P. Harrowell, J. Chem. Phys. 95 (1991) 4454.
[11] M. Schulz, P. Reineker, Phys. Rev. B 48 (1993) 9369.
[12] A. Heuer, U. Tracht, H.U. Spiess, J. Chem. Phys. 107 (1997) 3831.
[13] D. Stauffer, Int. J. Mod. Phys. C 11 (2000) 1081.
[14] S. Lawrence, C.L. Giles, Nature 400 (1999) 107.
[15] A. Kirman, J. Evol. Econ. 4 (1997) 339.
[16] S. Redner, Euro. Phys. J. B 4 (1998) 131.
[17] M. Schulz, S. Trimper, Phys. Rev. E 57 (1998) 6398.
[18] M. Schulz, P.R. Srikanth Sharma, H.L. Frisch, Phys. Rev. B 52 (1995) 7195.
[19] M. Doi, J. Phys. A 9 (1976) 1465.
[20] M. Doi, J. Phys. A 9 (1976) 1479.
[21] S. Sandow, S. Trimper, Eurosphys. Lett. 21 (1993) 799.
[22] G. Schütz, S. Sandow, Phys. Rev. E 49 (1994) 2726.
[23] M. Schulz, S. Trimper, J. Phys. A: Math. Gen. 29 (1996) 6543.
[24] C. Pigorsch, J.C. Kimball, H.L. Frisch, Phys. Rev. E 59 (1999) 3196.
[25] D. Stauffer, A. Aharony, Introduction to Percolation Theory, 2nd ed., Taylor and Francis, London, 1994.
[26] A.C.D. van Enter, J. Stat. Phys. 48 (1988) 943.
[27] J. Adler, A. Aharony, J. Phys. A 21 (1988) 1387.
[28] J. Adler, Physica A 171 (1991) 453.